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Unified analytical approach to the Darcy mixed convection with viscous dissipation in a vertical channel

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Abstract

The present paper is concerned with mixed convection in a vertical plane parallel channel filled with a porous medium. Steady parallel flow is examined, assuming that the effect of viscous dissipation is significant. Under these assumptions, the governing balance equations admit a first integral, so that the general solution can be given in an exact analytical form in terms of the Weierstrass' elliptic P-function. Based on this general solution and on a suitable parametrization of the problem, a unified approach which applies to all the thermal boundary conditions compatible with the steady parallel flow regime is reported. It is shown that the velocity field can either be unidirectional or bidirectional. Moreover, bidirectional flow configurations are possible also for vanishing average velocity, $U_m = 0$. A remarkable feature of the problem is that for $U_m < U_{m,\max}$, even two solution branches (*dual solutions*) exist, which merge when U_m approaches its maximum value $U_{m,\max}$. The general features of the solution space, as well as the mechanical and thermal characteristics of the flow are discussed for two cases of physical and engineering interest (*isoflux÷variable temperature*, and *isoflux÷isoflux* wall conditions) in some detail.

Keywords: Mixed convection; Viscous dissipation; Laminar flow; Fully developed regime; Analytical solution

1. Introduction

The current research interest in the fluid flow and heat transfer in porous media, as being documented in several comprehensive works published recently (see e.g. Nield and Bejan [1], Vafai [2], Bejan et al. [3], and Pop and Ingham [4]), is motivated by numerous applications of this class of phenomena in the modern technologies. Due to their important applications in mechanical, electrical, chemical, energy, environmental and civil engineering, a special attention has been paid to the internal flows in ducts and channels filled with porous media. The thermally developing forced convection flow in a parallel-plate channel or circular tube filled by a saturated porous medium with walls at uniform temperature or uniform heat flux, with axial conduction and viscous dissipation has been investigated by an extended Graetz method in a series of papers by Nield et al. [5–8] and Kuznetsov et al. [9]. An exact analytical solution of the Graetz problem for these basic duct geometries when the axial conductivity is significant has very recently been reported by Minkowycz and Haji-Sheikh [10]. The heat transfer in the thermal entrance region of a rectangular passage has been studied by Haji-Sheikh et al. [11] with the aid of the Green function method. Heterogeneity and variable viscosity effects in ducts filled with porous materials have been considered by Nield and Kuznetsov [12] and by Narashima and Lage [13] respectively. The mixed convection in narrow vertical ducts without the effect of viscous dissipation has been investigated by Pop et al. [14]. Storesletten and Pop [15] have extended the problem of buoyancy-driven viscous flow in a vertical parallel plane channel posed by Banks and Zaturska [16] to the case of a vertical porous layer with non-uniform wall temperature. The effect of viscous dissipation has been included in the study of the combined free and forced convection in a porous medium between two vertical walls by Ingham et al. [17]. More recent

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Nomenclature

A, B	dimensional constants	°C
C, D	dimensional constants of integration	°C
c_p	specific heat at constant pressure	$\mathrm{J\,kg^{-1}\,K^{-1}}$
E	dimensionless constant of integration	
g	gravitational acceleration	$\mathrm{m\,s^{-2}}$
g_2, g_3	invariants of the Weierstrass' elliptic function, Eqs. (29)	
k	thermal conductivity of the porous medium	$\mathrm{W\,m^{-1}\,K^{-1}}$
K	permeability of the porous medium	$\mathrm{m^2}$
L	width of the channel	m
P	hydrodynamic pressure	$\mathrm{kg\,m^{-1}\,s^{-2}}$
$\wp$	Weierstrass' elliptic function, Eq. (28)	
$\bar{q}$	heat flux, Eq. (6)	$\mathrm{W\,m^{-2}}$
$\bar{q}_{\text{frictional}}$	heat flux due to viscous dissipation, Eq. (13)	$\mathrm{W\,m^{-2}}$
q	dimensionless heat flux, Eq. (17)	
Ra	Rayleigh number for the porous medium, Eq. (21)	
T	temperature	°C
T_m	average temperature, Eq. (4b)	°C
T_{ref}	reference temperature	°C
T_*	temperature scale, Eq. (16)	°C
u	dimensionless velocity along the channel, Eq. (17)	
u_m	dimensionless average velocity, Eq. (17)	
U	dimensional velocity along the channel	$\mathrm{m\,s^{-1}}$
U_m	dimensional average velocity, Eq. (4a)	$\mathrm{m\,s^{-1}}$
U_*	velocity scale, Eq. (16)	$\mathrm{m\,s^{-1}}$
x, y	dimensionless axial and transversal coordinates	
X, Y	dimensional axial and transversal coordinates	m
y_0	dimensionless constant of integration, Eq. (37)	

Greek symbols

α	thermal diffusivity	$\mathrm{m^2\,s^{-1}}$
β	coefficient of thermal expansion	$\mathrm{K^{-1}}$
γ, λ	dimensionless parameters defined by Eq. (32)	
μ	dynamic viscosity	$\mathrm{kg\,m^{-1}\,s^{-1}}$
ρ	density	$\mathrm{kg\,m^{-3}}$
θ	dimensionless temperature, Eq. (19)	
ν	kinematic viscosity	$\mathrm{m^2\,s^{-1}}$

contributions to the effect of viscous dissipation in addition to the buoyancy effects have been published by Nield [18,19], and by Magyari et al. [20]. Analytical Taylor series solutions have been reported for the mixed convection in a vertical channel for isoflux÷isothermal wall conditions by Barletta et al. [21]. The same approach has been applied to the mixed convection channel flow of clear fluids for the case of symmetrical isothermal÷isothermal wall conditions by Barletta et al. [22].

The present paper revisits the problem of the fully developed mixed convection with non-negligible viscous dissipation in a vertical channel filled with a porous medium and reports a unified analytical approach which applies to all the thermal boundary conditions compatible with the steady parallel flow regime. This unified approach is based on the general solution of the governing balance equations which is obtained in an exact analytical form in terms of the Weierstrass' elliptic P-function. In order to be specific, for two cases of physical and engineering interest (*isoflux÷variable temperature*, and *isoflux÷isoflux* wall conditions), the features of the solution space, as well as the mechanical and thermal characteristics of the flow are discussed in detail.

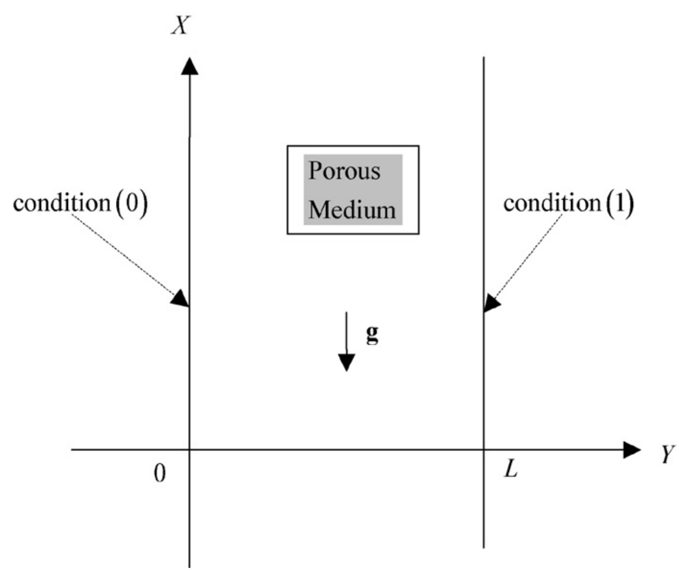


Fig. 1. Coordinate system and some general boundary conditions (0) and (1) at $Y = 0$ and $Y = L$, respectively.

2. Governing equations

2.1. Problem formulation

Consider mixed convection flow in a vertical parallel plane channel of width L, filled with a porous medium (Fig. 1). The vertical X-axis points opposite to the acceleration due to the gravity, $\mathbf{g}$. The Y-axis is perpendicular to the walls which are assumed to be impermeable. At each of the walls, a *thermal* boundary condition either of the first kind (a temperature distribution), or one of the second kind (a heat flux) will be prescribed. At the moment we name these prescriptions at $Y = 0$ and $Y = 1$ simply *condition* (0) and *condition* (1), respectively, and specify them later in detail. It is further assumed that the Darcy law, the Boussinesq and the Morton approximations hold, and that the heat generation by viscous friction is non-negligible. We examine steady parallel flow in which the only non-vanishing component of the seepage velocity field is its longitudinal component U (X-component). We first assume that this steady flow regime does really exist, and then specify the subsidiary conditions of its existence. Under these assumptions and, as a consequence of the continuity equation, the velocity U depends only on the transversal coordinate Y. The hydrodynamic pressure gradient in the transversal direction, $\partial P/\partial Y$, is vanishing, which means that P may depend only on X. Thus, the corresponding Darcy and energy balance equations read

$$\frac{\mu}{K}U = -\frac{\mathrm{d}P}{\mathrm{d}X} + \rho g\beta(T - T_{\mathrm{ref}}) \tag{1}$$

$$U\frac{\partial T}{\partial X} = \alpha\left(\frac{\partial^2 T}{\partial Y^2} + \frac{\partial^2 T}{\partial X^2}\right) + \frac{\nu}{Kc_p}U^2 \tag{2}$$

In the above equations, K is the permeability of the porous medium, g is the magnitude of the gravitational acceleration, β is the coefficient of thermal expansion, c_p is the specific heat at constant pressure, $\alpha = k/(\rho c_p)$ is the thermal diffusivity and k the thermal conductivity of the porous medium, ρ is the fluid density, μ is the dynamic viscosity, $\nu = \mu/\rho$ is the kinematic viscosity, $P = p + \rho g X$ is the hydrodynamic pressure. As required by the Boussinesq approximation, all the thermophysical properties are evaluated at the reference temperature T_{ref}. We assume that T_{ref} may depend on the wall coordinate X, $T_{\mathrm{ref}} = T_{\mathrm{ref}}(X)$. Thus, integrating Eq. (1) with respect to Y from 0 to L we get

$$\frac{\mu}{K}U_m = -\frac{\mathrm{d}P}{\mathrm{d}X} + \rho g\beta\big[T_m(X) - T_{\mathrm{ref}}(X)\big] \tag{3}$$

where

$$U_m = \frac{1}{L}\int_0^L U(Y)\,\mathrm{d}Y, \qquad T_m(X) = \frac{1}{L}\int_0^L T(X,Y)\,\mathrm{d}Y \tag{4a,b}$$

are average fluid velocity and temperature in a transversal section of the channel, respectively.

Subtracting Eq. (3) from Eq. (1), we obtain for the temperature field the expression

$$T(X,Y) = T_m(X) + \frac{\nu}{g\beta K}\big[U(Y) - U_m\big] \tag{5}$$

In Eq. (5), the thermophysical properties (which, as already mentioned, have to be taken at $T = T_{\mathrm{ref}}(X)$), also depend on X. In this respect the Morton approximation, [23] (an extension of the Boussinesq approximation) will be adopted, which assumes that the X-dependence of thermophysical properties may be neglected, and takes their values at some fixed station of the flow (e.g., at $X = 0$). Thus, Eq. (5) implies that the temperature field $T(X,Y)$ *decomposes additively* in an X- and an Y-dependent part.

The first important consequence of this additive decomposition of $T(X,Y)$ is that the transversal heat flux

$$\bar{q}(Y) = -k\frac{\partial T}{\partial Y} = -\frac{\nu k}{g\beta K}\frac{\mathrm{d}U}{\mathrm{d}Y} \tag{6}$$

is everywhere independent of the wall coordinate X. This feature imposes certain restrictions on the flux-boundary conditions which are compatible with the fully developed flow regime (see below).

A second important consequence of the additive decomposition of $T(X,Y)$ can be extracted from equation

$$U\frac{\mathrm{d}T_m}{\mathrm{d}X} - \alpha\frac{\mathrm{d}^2 T_m}{\mathrm{d}X^2} = \frac{\alpha\nu}{g\beta K}\frac{\mathrm{d}^2 U}{\mathrm{d}Y^2} + \frac{\nu}{Kc_p}U^2 \tag{7}$$

which has been obtained by substituting Eq. (5) in the energy equation (2). Indeed, the main message of Eq. (7) is that a fully developed channel flow with a *non-uniform* velocity profile $U = U(Y)$ only can exist when the average temperature T_m is a constant or, at most, a linear function of X,

$$T_m(X) = A\frac{X}{L} + B \quad \big[U = U(Y)\big] \tag{8}$$

where A and B (yet unknown) constants. In this case $U = U(Y)$ satisfies the equation

$$\frac{\mathrm{d}^2 U}{\mathrm{d}Y^2} + \frac{\rho g\beta}{k}U^2 - \frac{g\beta K A}{\alpha\nu L}U = 0 \tag{9}$$

and Eq. (5) of the temperature field becomes

$$T(X,Y) = A\frac{X}{L} + B + \frac{\nu}{g\beta K}\big[U(Y) - U_m\big] \tag{10}$$

Moreover, Eq. (7) also shows that a fully developed *slug flow* $U = constant = U_m$ is also possible when

$$U_m\frac{\mathrm{d}T_m}{\mathrm{d}X} - \alpha\frac{\mathrm{d}^2 T_m}{\mathrm{d}X^2} = \frac{\nu}{Kc_p}U_m^2 \tag{11}$$

In this case, Eq. (5) implies $T(X,Y) = T_m(X) \equiv T(X)$, which shows that the temperature field of the mixed convection slug flow depends only on the wall coordinate X, being given by the general solution of Eq. (11). This solution reads

$$T(X) = \frac{\nu U_m}{Kc_p}X + C\mathrm{e}^{\frac{U_m}{\alpha}X} + D \quad (U = U_m) \tag{12}$$

where C and D are constants of integration.

The above results allow us to specify all those thermal boundary conditions of the first and second kind which are compatible with the non-uniform parallel flow assumption adopted in the present paper. Indeed, in this respect Eqs. (10) and (6) show that the non-uniform velocity profile $U = U(Y)$ governed by Eq. (9) is compatible only with:

(i) a prescribed wall temperature which is a linear function of X (case $A \neq 0$), or
(ii) a prescribed constant wall temperature (case $A = 0$), or
(iii) a prescribed constant wall heat flux.

Obviously, at the two walls of the channel, all the combinations of the above conditions (and even their self-combinations) are allowed, except for the combination of (i) and (ii), which are mutually excluding conditions. It is worth underlining here again that, in contrast to the variable wall temperature conditions (i), no variable wall heat flux conditions are compatible with the fully developed flow. In the case of the slug flow $U = U_m$, where the transversal heat flux (6) is identically vanishing, even the constant wall heat flux conditions are excluded, except for the adiabatic conditions $\bar{q}(0) = \bar{q}(L) = 0$ (insulated channel). There are different variable or constant surface temperature conditions allowed, which can be obtained from Eq. (11) for specified values of C and D. All these cases of the slug flow in an adiabatically insulated channel are not a further concern of the present paper.

In addition to the velocity and temperature field, another quantity of a basic physical interest is the heat flux resulting from the volumetric heat generation by viscous friction,

$$\bar{q}_{\text{frictional}} = \frac{\mu}{K}\int_0^L U^2\,\mathrm{d}Y \tag{13}$$

A further basic relationship is the balance equation of the heat fluxes,

$$\bar{q}_{\text{frictional}} + \left[\bar{q}(0) - \bar{q}(L)\right] = \frac{kA}{L}\frac{LU_m}{\alpha} \tag{14}$$

which has been obtained by integrating Eq. (9) with respect to Y from 0 to L and taking into account Eqs. (6) and (13). This balance equation shows that, as expected, the average flux of the heat transported by the moving fluid (the right-hand side of Eq. (14)) equals the heat flux due to the viscous friction, added to the net heat flux through the walls of the channel.

It is worth emphasizing here that (owing to the Morton approximation) all the above results are independent of the choice of the reference temperature T_{ref}, except for Eq. (3) which connects the hydrodynamic pressure gradient $\mathrm{d}P/\mathrm{d}X$ to U_m, T_m and T_{ref}. A simple inspection of Eq. (3) suggests that it is advantageous to choose the reference temperature equal to the average temperature (8) since for $T_{\text{ref}} = T_m$, Eq. (3) reduces to the simple form

$$-\frac{\mathrm{d}P}{\mathrm{d}X} = \frac{\mu}{K}U_m \tag{15}$$

Eq. (15) shows that for the choice $T_{\text{ref}} = T_m$, the longitudinal pressure gradient $\mathrm{d}P/\mathrm{d}X$ is a constant quantity. A further desirable effect of the choice of $T_{\text{ref}} = T_m$ is that it maximizes the accuracy of the Boussinesq approximation, by minimizing the averaged square deviation from the local temperature of the fluid.

2.2. Nondimensionalization

For the subsequent calculations it is convenient to introduce the velocity and temperature scales

$$U_* = \frac{k}{\rho g\beta L^2}, \qquad T_* = \frac{\nu k}{\rho g^2\beta^2 L^2 K} = \frac{\nu U_*}{g\beta K} \tag{16}$$

as well as the dimensionless quantities

$$y = \frac{Y}{L}, \quad x = \frac{X}{L}, \quad u(y) = \frac{U(Y)}{U_*}, \quad u_m = \frac{U_m}{U_*}$$
$$q(y) = \frac{L}{kT_*}\bar{q}(Y), \quad q_{\text{frictional}} = \frac{L}{kT_*}\bar{q}_{\text{frictional}} \tag{17}$$

In terms of the dimensionless variables (19), Eqs. (9), (10) and (6) become

$$u'' + u^2 - Ra\,u = 0 \tag{18}$$

$$\frac{T(X,Y) - T_m(X)}{T_*} = u(y) - u_m \equiv \theta(y) \tag{19}$$

$$q(y) = -u'(y) \tag{20}$$

where the primes denote differentiations with respect to y and

$$Ra = \frac{g\beta KLA}{\alpha\nu} \tag{21}$$

is the Darcy–Rayleigh number. We mention that in connection with the mixed convection porous channel flows the dimensionless group (21) has first been used by Ingham et al. [17].

Furthermore, Eqs. (4a), (13), and (14) go over in

$$u_m = \int_0^1 u\,\mathrm{d}y \tag{22}$$

$$q_{\text{frictional}} = \int_0^1 u^2\,\mathrm{d}y, \tag{23}$$

$$q_{\text{frictional}} + q(0) - q(+1) = Ra\,u_m \tag{24}$$

Eqs. (10), (16), (17) and (20) give for the dimensional surface temperature distributions and the dimensionless surface heat fluxes the following expressions

$$\frac{T(X,0) - (Ax+B)}{T_*} = u(0) - u_m, \quad q(0) = -u'(0)$$
$$\frac{T(X,L) - (Ax+B)}{T_*} = u(1) - u_m, \quad q(1) = -u'(1) \tag{25}$$

According to Eqs. (21) and (25) the Darcy–Rayleigh number is non-vanishing only when both wall temperatures are (linear) functions of x, i.e. $A \neq 0$. The sign of A also carries an important physical information concerning the interplay of the external driving force with the buoyancy forces (aiding or opposing flows). On this reason, the behaviour of the solutions of our basic Eq. (18) under the sign change $Ra \to -Ra$ of the Darcy–Rayleigh number is a physically important feature of the problem. In this respect it can be shown that when $u(y, Ra; u_m)$ is a solution of Eq. (18) corresponding to the prescribed value u_m of the average velocity, then it also possesses the property

$$u(y, Ra; u_m) = Ra + u(y, -Ra; u_m - Ra) \tag{26}$$

Concerning the solution space of Eq. (18) it is also worth noticing that, in addition to the trivial solution $u = 0$, Eq. (18) admits the uniform solution $u = Ra = u_m$. The corresponding dimensionless temperature $\theta(y)$ given by Eq. (19) is identically vanishing, so that $T = T_m = Ax + B$ and $\bar{q} \equiv 0$.

3. The unified approach

3.1. The general analytical solution

Our basic differential equation (18) admits an exact analytical solution in terms of Weierstrass' elliptic function $P(y) = P(y; g_2, g_3)$ (for the properties of $P(y; g_2, g_3)$ see e.g. [24], Chapter 18). Indeed, one immediately sees that Eq. (18) possesses the first integral

$$\frac{1}{2}u'^2 + \frac{1}{3}u^3 - \frac{Ra}{2}u^2 = E \tag{27}$$

where E is an integration constant. Substituting $u(y) = (Ra/2) - 6\wp(y)$ in Eq. (27), we obtain for $\wp(y)$ precisely the differential equation of Weierstrass' $\wp$ function,

$$\wp'^2 = 4\wp^3 - g_2\wp - g_3 \tag{28}$$

with

$$g_2 = \frac{Ra^2}{12} \quad \text{and} \quad g_3 = -\frac{1}{18}\left(E + \frac{Ra^3}{12}\right) \tag{29a,b}$$

Eq. (28) is invariant under any translation $y \to y + y_0$ of the independent variable y. Thus, its general solution is of the form $\wp = \wp(y + y_0; g_2, g_3)$, where y_0 is the second constant of integration of the problem. Accordingly, the general solution for the velocity field has the form

$$u(y) = \frac{Ra}{2} - 6\wp(y + y_0; g_2, g_3) \tag{30}$$

and the corresponding average velocity is obtained as

$$u_m = \frac{Ra}{2} - 6\int_0^1 \wp(y + y_0; g_2, g_3)\,\mathrm{d}y \tag{31}$$

The general solution (30) of Eq. (18) involves in addition to the integration constants E and y_0, the Darcy–Rayleigh number (21) which, as long as the constant A has not been specified, is also an unknown quantity. Accordingly, the velocity problem with three unknown constants and only two wall conditions is still underdefined. On this reason, in addition to two boundary conditions, also the value U_m of the average flow velocity will be prescribed which, as being the volumetric flow rate through the transversal section of the channel, is an experimentally well accessible quantity. In turn, the prescription of U_m specifies via Eq. (15) also the value of the constant pressure gradient $\mathrm{d}P/\mathrm{d}X$ (which is not a priori known).

3.2. Parametrization

As mentioned in Section 2.1, at the two walls of the channel several combinations and self-combinations of the boundary conditions (i)–(iii) may be prescribed. Thus, the desire for a unified description which applies to all these cases of practical interest arises in a natural way. The general solution (30) being known, the remaining task is to find such a parametrization of the problem which allows us to determine the constants of integrations E and y_0 when any two of the compatible wall conditions (i)–(iii) are prescribed.

Our option for this procedure is the parametric prescription at the left wall ($Y = 0$) of the velocity $u(0)$ and of the heat flux $q(0) = -u'(0)$. For these working parameters, the following short notations will be used

$$u(0) = \gamma, \qquad u'(0) = -\lambda \tag{32}$$

The main reason for the parametrization (32) is that Eq. (18) along with the conditions (32) specifies an initial value problem which, as it is well known, admits a unique solution for any given values of γ and λ. Thus, the constants of integrations E and y_0 will be determined in terms of Ra, γ and λ uniquely, and the solution of the initial value problem (18), (32) may be written in the form $u = u(y, Ra, \gamma, \lambda)$. Indeed, according to this procedure, Eqs. (27) and (32) yield for E the explicit expression

$$E = \frac{1}{2}\lambda^2 + \frac{1}{3}\gamma^3 - \frac{Ra}{2}\gamma^2 \tag{33}$$

while Eqs. (32) and (30) result in

$$\wp(y_0; g_2, g_3) = \frac{Ra - 2\gamma}{12} \tag{34}$$

$$\wp'(y_0; g_2, g_3) = \frac{\lambda}{6} \tag{35}$$

With the aid of Eq. (33), the integration constant E can be eliminated from Eq. (29b) of g_3, so that

$$g_3 = -\frac{1}{36}\left(\lambda^2 + \frac{2}{3}\gamma^3 - Ra\,\gamma^2 + \frac{Ra^3}{6}\right) \tag{36}$$

Furthermore, it is known (see e.g. [25], p. 1173), that the system of equations $\{\wp(z; g_2, g_3) = p, \wp'(z; g_2, g_3) = s\}$ admits for z a unique solution in terms of the inverse of Weierstrass' $\wp$ function when for p and s the relationship $s^2 = 4p^3 - g_2 p - g_3$ holds. This solution is $z = \mathrm{InverseWeierstrass}\,\wp[\{p, s\}, \{g_2, g_3\}]$. In the case of our Eqs. (34) and (35), we have $p = (Ra - 2\gamma)/12$, $s = \lambda/6$, $g_2 = Ra^2/12$, and g_3 is given by Eq. (36). It is easy to show that in this case the condition $s^2 = 4p^3 - g_2 p - g_3$ is satisfied identically, so that

$$y_0 = \mathrm{InverseWeierstrass}\,\wp\left[\left\{\frac{Ra - 2\gamma}{12}, \frac{\lambda}{6}\right\}, \{g_2, g_3\}\right] \tag{37}$$

Therefore, the constants of integrations E and y_0 are uniquely determined in terms of Ra, γ and λ and thus the integral condition (31) expresses u_m as a function of Ra, γ and λ. Then, the two boundary conditions yield two additional equations for Ra, γ, λ. Accordingly, the parameters Ra, γ, λ can be expressed in terms of u_m and the two surface quantities prescribed by the boundary conditions. In this way the velocity problem has basically been solved and the subsequent task is to determine the values of Ra, γ and λ from Eq. (31) and the two wall conditions selected from the four equations (25). In order to be specific, in the next two sections we show how this general procedure works, by applying it to some boundary conditions which, to our knowledge, have not yet been investigated in the literature.

4. Applications

4.1. Isoflux÷variable temperature wall conditions

As a first application of the solution the procedure described in Section 3, we consider the case when the *conditions* (0) and (1) indicated in Fig. 1 are selected from the four equations (25) as follows:

$$\textit{Condition } (0){:}\quad q(0) = q_0 \tag{38a}$$

$$\textit{Condition } (1){:}\quad T(X, L) = T_1 x + T_2 \tag{38b}$$

The constants q_0, T_1 and T_2 occurring in Eqs. (38) are given. According to the classification described in Section 2.1, the *isoflux* and the *variable wall temperature* conditions (38) correspond to the *hybrid* combination (iii)–(i), of a boundary condition of the second and of the first kind, respectively. Now, substituting Eq. (38b) into the third equation (25) and then comparing Eq. (38a) to the second equation (25) and to Eqs. (32), we obtain

$$A = T_1, \quad B = T_2 - T_*\big[u(1) - u_m\big], \quad \lambda = q_0 \tag{39}$$

The first equation (39) specifies the value of Ra by Eq. (21), and the third equation (39) sets the parameter λ equal to the prescribed wall heat flux q_0, so that the velocity solution $u = u(y, Ra, \gamma, \lambda) = u(y, Ra, \gamma, q_0)$ given by Eq. (30) contains a single free parameter, namely the parameter γ, only. The value of γ can then be determined for a prescribed u_m from the integral condition (31). Thus, the velocity solution $u = u(y; Ra, \gamma, q_0)$ being known, the second equation (39) gives the value of the constant B, so that the average temperature $T_m(X)$ is determined by Eq. (8) explicitly. The dimensional temperature field is obtained subsequently from Eq. (10). In this way, the problem corresponding to the hybrid boundary conditions (38) has basically been solved, and the only remaining "detail" is to solve Eq. (31) for γ when the values of Ra, q_0 and u_m are specified.

In order to gain a deeper insight in the latter issue, in Fig. 2 the average velocity as a function of γ has been plotted according to Eq. (31) for $q_0 = 1$ and three different values of the Darcy–Rayleigh number Ra.

Fig. 2 emphasizes the following properties of the function $u_m = u_m(Ra, \gamma, q_0)$.

- For specified Ra and q_0, solutions only exist when the prescribed value of u_m does not exceed a maximum value $u_{m,\max}$, i.e., the domain of existence of the solution is $u_m \leqslant u_{m,\max}$, where $u_{m,\max}$ is reached at a value $\gamma \equiv \gamma_{\max}$ of the parameter γ which depends on Ra and q_0, $\gamma_{\max} = \gamma_{\max}(Ra, q_0)$.

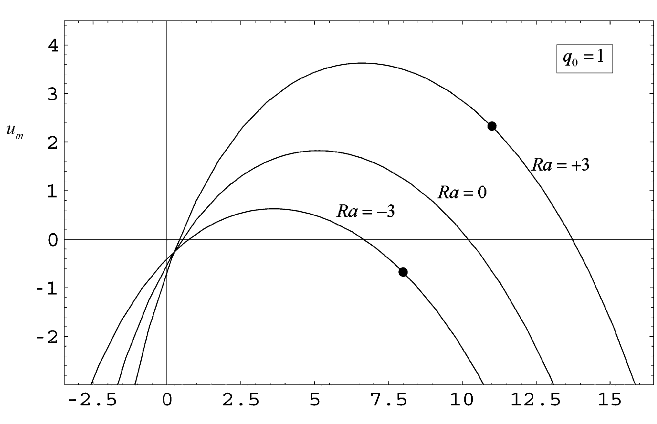


Fig. 2. Plot of the average velocity (31) as a function of the parameter γ, for $q_0 = 1$ and three different values of Ra with increasing values of Ra, the maximum value $u_{m,\max}$ of u_m increases and, at the same time, moves to the right. The dots on the curves corresponding to $Ra = +3$ and $Ra = -3$ have the coordinates $(\gamma, u_m) = (11, 2.323645)$ and $(\gamma, u_m) = (8, -0.676355)$, respectively, and are related to each other according to Eq. (40).

- The existence domain $u_m \leqslant u_{m,\max}$ is not bounded from below; all negative values of u_m being allowed.
- For any prescribed value of u_m in the range $u_m < u_{m,\max}$, the mixed convection problem admits two solutions (*dual solutions*) corresponding to two different values $\gamma_1(u_m)$ and $\gamma_2(u_m)$ which become coincident for $u_m = u_{m,\max}$, i.e. $\gamma_1(u_{m,\max}) = \gamma_2(u_{m,\max}) = \gamma_{\max}(Ra, q_0)$.
- With increasing values of Ra (and a specified q_0), the maximum $u_{m,\max}$ increases and, at the same time, moves to the right. As a consequence of the symmetry property (26), the following relationships hold

$$u_m(Ra, \gamma, q_0) = Ra + u_m(-Ra, \gamma - Ra, q_0) \tag{40}$$

$$\gamma_{\max}(Ra, q_0) = Ra + \gamma_{\max}(-Ra, q_0) \tag{41}$$

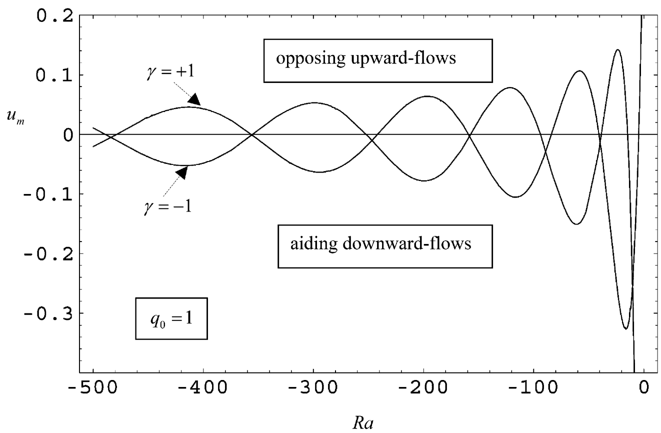
$$u_{m,\max}\big[\gamma_{\max}(Ra, q_0)\big] = Ra + u_{m,\max}\big[\gamma_{\max}(-Ra, q_0)\big] \tag{42}$$

In the cases $(Ra, q_0) = (3, 1)$ and $(Ra, q_0) = (-3, 1)$ shown in Fig. 2, we find $\gamma_{\max}(3, 1) = 6.61467$, $\gamma_{\max}(-3, 1) = 3.61467$, $u_{m,\max}[\gamma_{\max}(3, 1)] = 3.62616$ and $u_{m,\max}[\gamma_{\max}(-3, 1)] = 0.62616$, in full agreement with Eqs. (41) and (42). The two dots of Fig. 2 are related to each other according to Eq. (40) for $Ra = 3$ and $\gamma = 11$, in which case $u_m(-3, 8, 1) = -0.676355$ and $u_m(3, 11, 1) = 3 - 0.676355 = 2.323645$.
- Nontrivial dual flow solutions of vanishing average velocity, $u_m = 0$, also can exist, as long as $u_{m,\max} \geqslant 0$. The integral balance equation (24) shows that in this case, no heat is transported by the moving fluid in the longitudinal direction. The sum of the incoming heat flux $q(0)$ and of the heat flux $q_{\text{frictional}}$ due to the viscous friction equals the outgoing heat flux $q(+1)$ through the right wall of the channel. The latter property holds obviously also for $u_m \neq 0$, but $Ra = 0$.

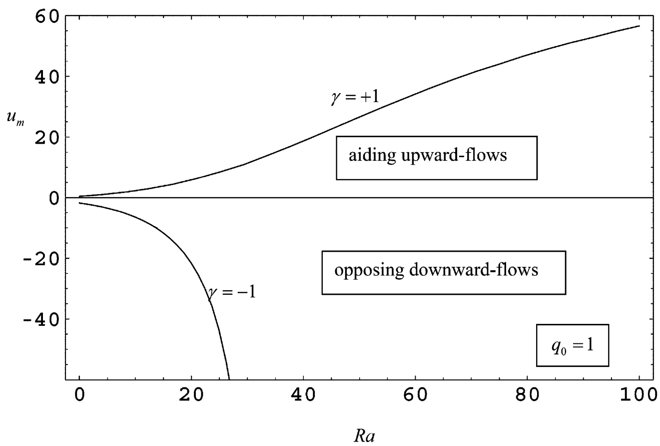
As already mentioned, the sign of Ra characterises the interplay between the external driving force and the buoyancy forces. Thus, the buoyancy aids the forced convection flow when $\operatorname{sgn} Ra = \operatorname{sgn} u_m$, and opposes it when $\operatorname{sgn} Ra = -\operatorname{sgn} u_m$. As an illustration of the quite complicated dependence of u_m on Ra, in Figs. 3(a), (b) the function $u_m = u_m(Ra, \gamma, q_0)$ has been plotted for $q_0 = 1$, $\gamma = +1$ and $\gamma = -1$ in the ranges $Ra \leqslant 0$ and $Ra \geqslant 0$, respectively. Having in mind that the viscous dissipation term of the energy equations breaks the physical up/down equivalence of the buoyant flows over and upward projecting hot plate and over its downward projecting cold counterpart (see [26]), in the present case the aiding upward-flows are physically distinct from the aiding downward-flows. The same holds for the opposing up- and downward-flows, too. These features are clearly seen in Figs. 3(a) and (b).

In addition to the velocity and temperature field, in the case of boundary conditions (38) the following two quantities are of engineering interest.

- The temperature distribution $T(X, 0)$ of the left wall of the channel, where the *incoming heat flux* $q(0)$ has been prescribed, and



(a)



(b)

Fig. 3. (a) Plots of the average velocity u_m as a function of *Ra* for $q_0 = 1$ and $\gamma = +1$ and $\gamma = -1$, in the range $Ra \leqslant 0$. The two curves are not symmetric with respect to the *Ra*-axis. At $Ra = 0$, e.g., one has $u_m = 0.405624$ and $u_m = -1.80747$ for $\gamma = +1$ and $\gamma = -1$, respectively. (b) Plots of the average velocity u_m as a function of *Ra* for $q_0 = 1$ and $\gamma = +1$ and $\gamma = -1$, in the range $Ra \geqslant 0$.

- The *outgoing heat flux* through the right wall of the channel, $q(1)$, where the temperature distribution $T(X, L) = T_1 x + T_2$ has been prescribed.

The explicit expressions of these quantities are obtained from Eqs. (25), (30), (34) and (20) as:

$$\frac{T(X,0) - T(X,L)}{T_*} = u(0) - u(1) \equiv \theta_w(0) = \frac{2\gamma - Ra}{2} + 6\wp(1 + y_0; g_2, g_3) \tag{43}$$

$$q(1) = 6\wp'(1 + y_0; g_2, g_3) \tag{44}$$

The expressions (29a) and (36) of g_2 and g_3 as well as the difference $2\gamma - Ra$ occurring in Eqs. (37) and (43) are invariant under the transformation $\{Ra \to -Ra;\ \gamma \to \gamma - Ra\}$. As a consequence the dimensionless temperature (43) of the left wall, as well as the dimensionless heat flux (44) through the right wall of the channel satisfy the simple symmetry relationships

$$\begin{aligned} \theta_w(0, Ra, \gamma, q_0) &= \theta_w(0, -Ra, \gamma - Ra, q_0) \\ q(1, Ra, \gamma, q_0) &= q(1, -Ra, \gamma - Ra, q_0) \end{aligned} \tag{45}$$

We may conclude that for the boundary conditions (38), all the quantities of physical and engineering interest can be can be calculated with the aid of the unified approach explicitly. In addition, it is worth emphasizing here the results corresponding to some important particular cases of the conditions (38), also can be recovered from the results of the present section. Such cases are (i) the *isoflux÷isothermal* boundary conditions corresponding to $q_0 \neq 0$ and $T_1 = 0$ (i.e. $Ra = 0$), and (ii) the *adiabatic÷isothermal* boundary conditions corresponding to $q_0 = 0$ and $Ra = 0$. Furthermore, the latter case ($q(0) = 0, T(X, L) = T_2$) yields the "half-channel" results corresponding to the symmetric *isothermal÷isothermal* boundary conditions $T(-L) = T(+L) = T_2$, prescribed for a channel of width $2L$, $Y \in [-L, L]$. Similarly, the case ($q_0 = 0, T(X, L) = T_1 x + T_2$) yields the "half-channel" results corresponding to the symmetric *variable temperature÷variable temperature* conditions) $T(X, -L) = T_1 x + T_2$, $T(X, +L) = T_1 x + T_2$) prescribed for a channel of width $2L$.

4.2. Isoflux÷isoflux wall conditions

We consider now the case in which the *conditions* (0) and (1) are selected from the four Eqs. (25) as

$$\textit{Condition } (0)\text{:} \quad q(0) = q_0 \tag{46a}$$

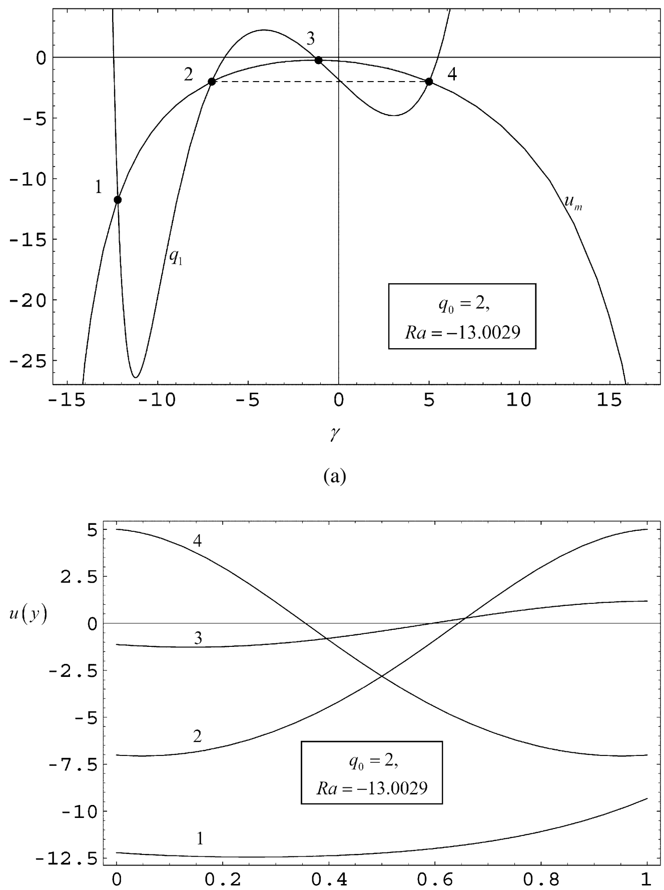
$$\textit{Condition } (1)\text{:} \quad q(1) = q_1 \tag{46b}$$

where the constant heat fluxes q_0, and q_1 are given. According to the classification described in Section 2.1, the conditions (46) correspond to the self-combination of the isoflux conditions (iii). The condition (46a) sets again parameter λ equal to the prescribed incoming wall heat flux q_0. Comparing to the *isoflux÷variable wall temperature* conditions (38), the essential difference consists now in the fact that the values of A and B occurring in Eqs. (25) are not determined by the isoflux condition (46b) explicitly. Instead, the value of A is determined via *Ra* and the equation

$$6\wp'(1 + y_0; g_2, g_3) = q_1 \tag{47}$$

implicitly, as a function of q_0, q_1 and γ, while B still remains unspecified. In this way, owing to Eq. (47), the velocity $u = u(y, Ra, \gamma, q_0)$ becomes a function of y, q_0, q_1 and γ (and the average velocity u_m a function of q_0, q_1 and γ). In other words, specifying the values of q_0, and q_1, Eq. (47) yields *Ra* as a function of γ, and Eq. (31) gives u_m as a function of γ (for the specified values of q_0, and q_1). Thus, for a prescribed value of u_m, the corresponding values of γ can be calculated, and the problem is basically solved again. As mentioned above, for the *isoflux÷isoflux* boundary conditions (46) the value of the integration constant B remains undetermined. This circumstance, however, is physically not disturbing since it corresponds to our freedom to choose the origin of the temperature scale at our will.

The solution space possesses a rich structure which is illustrated by Fig. 4(a) where q_1 and u_m, as being given by the respective equations (47) and (31), have been plotted as func-



(a)

(b)

Fig. 4. (a) Plot of q_1 and u_m as functions of γ for $q_0 = 2$, and $Ra = -13.0029$. (b) Plots of the dimensionless velocity profiles $u(y)$ associated with the four points of (a).

tions of γ for $q_0 = 2$, and $Ra = -13.0029$. The curves q_1 and u_m have in this case four intersection points, denoted by 1, 2, 3 and 4, respectively. The coordinates of these intersection points are $(\gamma, q_1 = u_m)_1 = (-12.2057, -11.7737)$, $(\gamma, q_1 = u_m)_2 = (-7.01347, -2)$, $(\gamma, q_1 = u_m)_3 = (-1.12100, -0.23704)$ and $(\gamma, q_1 = u_m)_4 = (5.00667, -2)$. The value $Ra = -13.0029$ has been chosen so that for the intersection points 2 and 4 (joined in Fig. 4 by a dashed line), in addition to $q_1 = u_m$, also the equality $(q_1 = u_m)_2 = (q_1 = u_m)_4 = -2$ holds. This means that the two corresponding solutions $u(y)$ obtained for $\gamma = \gamma_2 = -7.01347$ and $\gamma = \gamma_4 = 5.00667$ are associated with the same set of values of the three prescribed quantities,$(q_0, q_1 = u_m)_2 = (q_0, q_1 = u_m)_4 = (2, -2)$. For the intersection points 1 and 3, in contrast, the distinct sets of values $(q_0, q_1 = u_m)_1 = (2, -11.7737)$ and $(q_0, q_1 = u_m)_3 = (2, -0.23704)$ hold. The velocity profiles associated with the points 1, 2, 3 and 4 are plotted in Fig. 4(b). All the four velocity profiles describe downward aiding flows ($u_m < 0, Ra < 0$) but, while the flow 1 is unidirectional, the other three are bidirectional flows. Having in mind that, on account of Eq. (19), the dimensionless temperature profiles and $\theta(y)$ are shifted with respect to the velocity profiles $u(y)$ by $-u_m$, $\theta(y) = u(y) - u_m$, their average values are always identically vanishing,

$$\theta_m = \int_0^1 \theta \, dy = \int_0^1 (u - u_m) \, dy = 0 \tag{48}$$

5. Results and discussion

The main results of the paper can be summarized as follows.

- The governing differential equation of the velocity field admits the first integral

$$\frac{1}{2}u'^2 + \frac{1}{3}u^3 - \frac{Ra}{2}u^2 = \text{constant} \equiv E \tag{49}$$

  where u'^2 represents the square of the dimensionless heat flux $q(y) = -u'(y)$.
  Eq. (49) resembles the conservation law of the total mechanical energy during the 1D motion of a particle of mass $m = 1$, coordinate u, velocity u' and of potential energy $W = u^3/3 - Ra\,u^2/2$, the time variable being y. Similarly, Eq. (49) represents also in the present context a differential "conservation law". Its message is that the interplay between the driving external force (pressure gradient), buoyancy, viscous dissipation heat diffusion is such, that the effect of the boundary conditions is transferred from one wall of the channel to the other by preserving the value of the left-hand side of Eq. (49) for all distances y. In particular, the relationship

$$\left(\frac{1}{2}u'^2 + \frac{1}{3}u^3 - \frac{Ra}{2}u^2\right)_{\text{left wall}} = \left(\frac{1}{2}u'^2 + \frac{1}{3}u^3 - \frac{Ra}{2}u^2\right)_{\text{right wall}} \tag{50}$$

  holds. The practical benefit of Eq. (50) is that it connects the wall values of the heat flux $q(y) = -u'(y)$ and temperature $\theta(y) = u(y) - u_m$ across the channel to each other.
- The existence of the dual solutions and of the bidirectional flows with vanishing average velocity (properties, which have already been reported in previous investigations), turn out to represent general features of the solution space when in the mixed convection the (nonlinear) effect of the viscous dissipation is taken into account.
- A further general effect of the viscous dissipation is that it breaks the usual equivalence of the aiding upward and downward, as well as of the opposing upward and downward mixed convection flows, respectively (see Figs. 3). As being already emphasized by Al-Hadhrami et al. [27], all these aiding and opposing flows are physically realizable.
- The approach reported in the present paper for various combinations and self combinations of the thermal boundary conditions of the first and second kind, (i)–(iii) (listed in Section 2.1), can also be extended to the case of (linear) thermal boundary conditions of the third kind.

6. Conclusions

We have considered mixed convection in a vertical plane parallel channel filled with a porous medium. Steady parallel flow has been examined, assuming that the effect of viscous dissipation is significant. The main message of the paper is that the general solution of the governing balance equations can be given in an exact analytical form in terms of the Weierstrass' elliptic P-function. Based on this exact solution, a unified analytical approach could be developed which applies to all the thermal boundary conditions compatible with the steady parallel flow regime. As an illustration of this method, the cases of the *isoflux÷variable temperature*, and *isoflux÷isoflux* wall conditions have been worked out in detail.

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